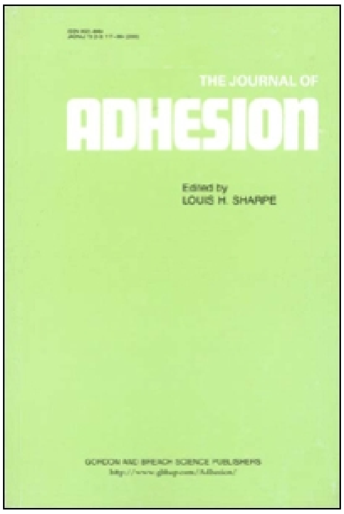

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Yield in Adhesive Joints and Design of Zones with Constant Elastic Shear Stresses

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Yield in adhesive joints has been investigated by several scientists among whom L. J. Hart-Smith[1] especially is to be mentioned.

In the following, a method is demonstrated which is based on a simple elastic-plastic model. It shows the distribution of stresses in the adhesive and gives a total picture of the development of the length of the yield zones and their strain as a function of load.

Methods are given for the design of adhesive joints with constant elastic shear stresses at their ends or throughout their whole length. These stresses are obtained by varying the thickness of the adherends, the adhesive, or a combination of both. The constant elastic shear stress zones can be designed to take into consideration all known factors as temperature and hardening stresses, moments, etc. The characteristic yield properties as well as internal stresses after yield and unloading are determined together with the modified stress distribution for a new load.

KEY WORDS Plastic deformation; variable thickness of adhesive and adherend; zones of constant elastic shear stress; adhesive joints; analysis; design; stresses after yield and load changes.

ELASTIC DEFORMATION

Elastic shear stresses in adhesive joints are often calculated by means of Volkersen's formula.[2] It offers a simplified picture that does not account for bending moments, and it assumes tensile stresses to be uniform across the sections of the adherends.

The following calculations based on Volkersen's formula are therefore valid with fairly good accuracy only for joints where bending moments are avoided, as in double-lap joints or reduced, as in cylindrical joints. For the latter, the thickness of an adherend shall be calculated as the sectional area of the adherend divided by the periphery of the cylinder. The accuracy is best for a solid inner adherend. For single lap-joints the figures are inaccurate, as the bending moments will give increased stresses at the ends of the joint; still they give a clear picture of the principal behaviour of the joint.

In the section "Accurate Calculation of Zones with Constant Shear Stresses" bending moments and other factors are considered, and the accuracy is high also for single-lap joints.

By rewriting Volkersen's well-known formula with new constants that are independent of the length of the adhesive joint, we get shear stresses at a point x of the adhesive joint determined by:

$$\tau = \frac{c_1 \cdot P}{\sinh(c_2 \cdot L)} \cdot [c_3 \cdot \cosh(c_2 x) + \cosh(c_2 \cdot (L - x))] \tag{1}$$

where

$$c_1 = \left(\frac{\Delta}{k}\right)^{1/2} : L$$

$$c_2 = (\Delta \cdot k)^{1/2} : L$$

$$c_3 = k - 1 = \frac{E_2 \cdot t_2}{E_1 \cdot t_1} \quad \text{(rigidity factor of the adherends)}$$

The most rigid adherend is denoted by 2 and, consequently, we always have $c_3 > 1$.

$$\Delta = \frac{G \cdot L^2}{E_2 \cdot t_2 \cdot t_a} \qquad k = 1 + \frac{E_2 \cdot t_2}{E_1 \cdot t_1}$$

with

- x length coordinate in the adhesive joint mm
- L length of the adhesive joint mm
- t_a adhesive thickness mm
- t_1 thickness of adherend 1 mm
- t_2 thickness of adherend 2 mm
- E_1 modulus of elasticity of adherend 1 N/mm^2
- E_2 modulus of elasticity of adherend 2 N/mm^2
- G modulus of shear of adhesive N/mm^2
- P load per mm of adhesive joint width N/mm.

The dimensions refer to Figure 1, which shows a single-lap joint. To avoid bending moments it can be assumed to be half a double-lap joint.

PLASTIC DEFORMATIONS

To enable a calculation to be made of plastic deformations in the adhesive joint, it is necessary to know the adhesive shear stress-strain curve which may have the appearance shown in Figure 2. The solid line is the actual stress-strain curve at the joint thickness and temperature involved. The dotted line indicates an

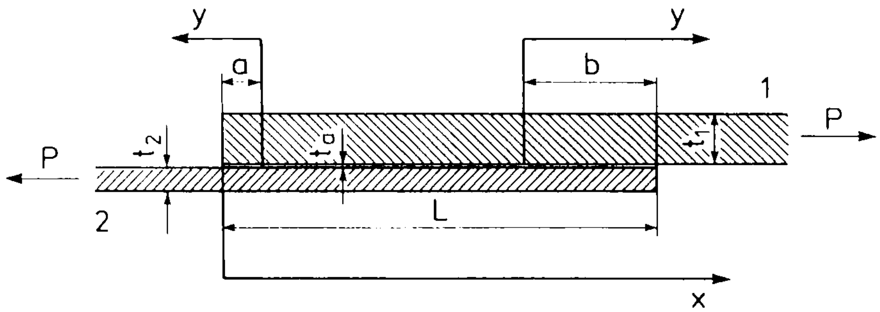

FIGURE 1 Adhesive joint.

approximation which will allow the characteristics of the adhesive joint to be calculated. The areas below the curves indicate the work capacity, and they are similar.

For adhesive joints such as those in Figure 1, Eq. (1) shows that the shear stresses are highest near the ends of the adhesive joint, and they might resemble those in Figure 3.

From the calculation of the stresses, we find that the stress distribution to the left of an arbitrary point O is determined by the shear forces at O, *i.e.* the outer force and the sum of the shear forces to the right of O, while the distribution of these shear forces is of no significance.

This means, for instance, that one could imagine the possibility of altering the shear stresses to the right of O to produce a constant value equal to the stress at O, and at the same time increase the length of the adhesive joint. This means that the total shear force to the right of O would remain unaltered, as shown by the unaltered magnitude of the area below the stress curve to the right of O; this is indicated by the dotted line in Figure 3.

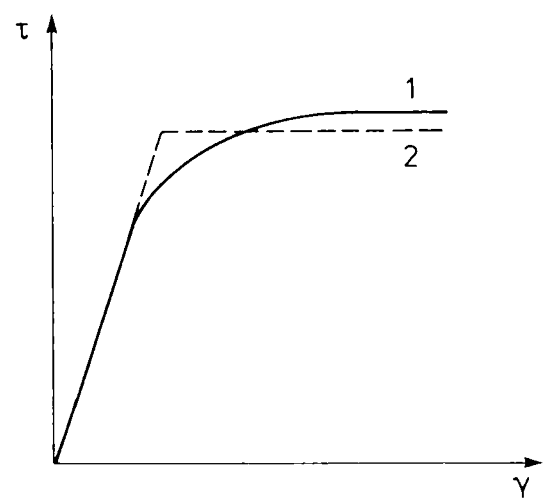

FIGURE 2 Actual and approximated stress-strain curve of adhesive.

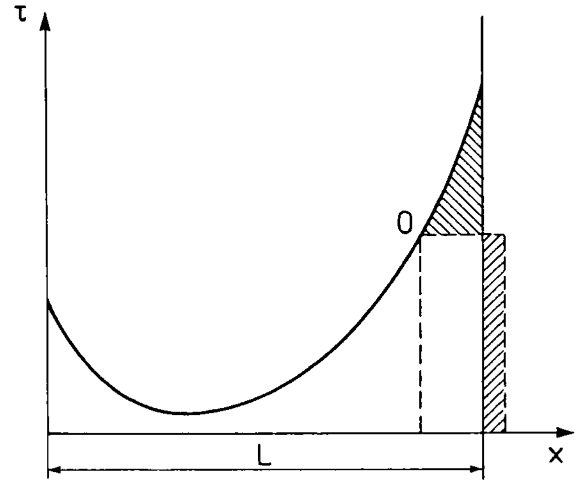

FIGURE 3 Stress distribution in adhesive joint and equivalent stress distribution with yield.

This figure corresponds to the presence of a constant yield stress to the right of O. The stresses to the left of O, however, are still being determined by Eq. (1), but the length of the adhesive joint is now different from the original one.

If we wish to establish constant stress on the original length of joint we must first apply it to an adhesive joint as base that is somewhat shorter than the original one. This is done because it will always be possible to choose a magnitude of the constant stress such that the length of the zone with constant stress and unchanged shear force in the connecting point will result in an adhesive joint with just the required initial length.

In many cases, yield stress is present at both ends of the adhesive joint and, if so, with equal constant stresses. The procedure will then require that we choose as base an adhesive joint with length L_u that is less than L. For an arbitrary force P_U, the stress curve is determined by Eq. (1) after replacing L with L_u and P with P_u.

We can see from Figure 4 that a value of the constant stresses $\tau_{bu} = \tau_{au}$ is now determined such that the two hatched areas become equal, and that the resultant extension of the adhesive joint results in exactly the length L. The constant tension τ_{bu} can be measured in the figure, and it normally will be different from the yield stress τ_f.

It appears from Eq. (1) that, at an arbitrary point in the adhesive joint, the stress τ is proportional to P. Consequently, we conclude that the constant stress τ_{bu} becomes equal to the yield stress τ_f by plotting a new curve for a force:

$$P = P_u \cdot \frac{\tau_f}{\tau_{bu}}$$

or by changing the scale for the stress correspondingly, while the scale for the position x is maintained unchanged. Thus we have determined the force P

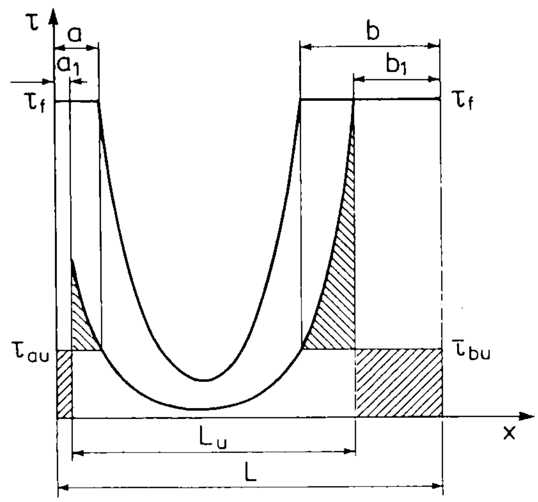

FIGURE 4 Determination of the constant stresses $\tau_{au} = \tau_{bu}$ at the ends of the adhesive joint L which gives the same load for L as shown for L_u. The top curve shows the stresses when τ_{au} and τ_{bu} are increased to τ_f.

resulting in yield stress in the zones a and b, as shown in the figure, which also gives the distances a_1 and b_1 between the end points L_u and L. At the same time, the total stress distribution throughout the adhesive joint of this force P is also being determined, as the varying stresses are found by Eq. (1) on inserting values for length L_u and force P.

After repeating these procedures only once with a new value for L_u, it will be possible to draw a curve as in Figure 5 for the connection between the force P

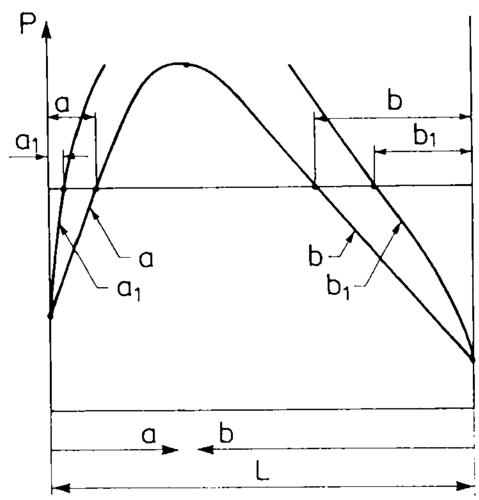

FIGURE 5 The correlation for a given adhesive joint between the load P and lengths a and b of zones with constant stress τ_f. These zones may be formed by yield or by a special design of the joint. a_1 and b_1 determine the location of L_u.

and the length of the yield zones a and b. Each value of L_u determines two points of the curve. The top point of the curve is given by $P = \tau_f \cdot L$ and $b:a = c_3$, as will be shown later. The low point is given directly by Eq. (1). In the same figure, curves are drawn for a_1 and b_1 to locate L_u.

Figure 5 shows the extension of the yield zones from the initial yield at one end *via* an increased yield at both ends to the yield throughout the adhesive joint. However, in many cases fractures will occur earlier because we reach the ultimate strain or a strain where the yield stress drops so much that the joint cannot hold.

We will determine the shear strain along b (Figures 1 and 5) using y as position coordinate with zero point to the left end of b where the shear angle and shear are, respectively:

$$\gamma = \frac{\tau_f}{G} \qquad \Delta y_D = \gamma \cdot t_a = \frac{\tau_f \cdot t_a}{G} \tag{2}$$

The force in adherend 2 will decrease linearly from $b \cdot \tau_f$ for $y = 0$ to 0 for $y = b$:

$$P_{2y} = \tau_f \cdot (b - y) \qquad \sigma_{2y} = \frac{\tau_f}{t_2} \cdot (b - y) \tag{3}$$

Assuming point $y = 0$ to be fixed, the displacement of y towards the right in adherend 2 will be:

$$\Delta y_2 = \int_0^y \frac{\sigma_{2y}}{E_2} \cdot dy = \frac{\sigma_f}{t_2 \cdot E_2} \int_0^y (b - y) dy = \frac{\sigma_f}{t_2 \cdot E_2} \cdot \left(by - \frac{y^2}{2}\right) \tag{4}$$

The force in adherend 1 will increase linearly from $P - \tau_f \cdot b$ for $y = 0$ to P for $y = b$:

$$P_{1y} = P - \tau_f \cdot (b - y) \qquad \sigma_{1y} = \frac{P - \tau_f(b - y)}{t_1} \tag{5}$$

$$\begin{aligned} \Delta y_1 &= \int_0^y \frac{1}{t_1 \cdot E_1} (P - \tau_f \cdot (b - y)) dy \\ &= \frac{1}{t_1 \cdot E_1} \left(P \cdot y - \tau_f \cdot b \cdot y + \tau_f \cdot \frac{y^2}{2}\right) \end{aligned} \tag{6}$$

The shear of adherend 1 towards the right relative to adherend 2 will then be:

$$\Delta y = \Delta y_D + \Delta y_1 - \Delta y_2 \tag{7}$$

$$\Delta y = \frac{\tau_f \cdot t_a}{G} + \frac{1}{t_1 \cdot E_1} \left(P \cdot y - \tau_f \left(by - \frac{y^2}{2}\right)\right) - \frac{\tau_f}{t_2 \cdot E_2} \left(by - \frac{y^2}{2}\right) \tag{8}$$

where Δy is the shear in the yield zone of the adhesive joint. The shear is highest at the end of adherend 2, where $y = b$:

$$\Delta y_b = \frac{\tau_f \cdot t_a}{G} + \frac{1}{t_1 \cdot E_1} \left(P \cdot b - \tau_f \cdot \frac{b^2}{2}\right) - \frac{\tau_f \cdot b^2}{t_2 \cdot E_2 \cdot 2} \tag{9}$$

The shear strain is $\gamma = \Delta y / t_a$.

It is unnecessary to calculate the shear strain at the other end of the adhesive joint, because with constant thickness of adhesive and adherends we will get the maximum strain at the end of the rigid adherend 2.

CONSTANT ELASTIC STRESSES

Our knowledge of yield stresses can now be utilized in 3 different ways to create constant elastic stresses at the ends of the adhesive joint or through the total length of the joint.

In Figure 5 the curve for P as function of a and b can be called the characteristic of the joint. Using this curve we can design zones with constant elastic stresses, and all stresses and strains in the joint for all loads and load variations with or without yield can be calculated. τ_f is the yield stress, but it may also be considered as the maximum attainable elastic stress appearing just before yield occurs.

If this constant stress τ_f is the result of the design of the adhesive joint and not caused by yield, all stresses, both in the adhesive and in the adherends, are elastic. Consequently, they are proportional to the load and zones of the joint which, at a given load, have constant stresses, will retain constant stresses at all other loads also, as long as no yield has occurred.

If yield has occurred the situation is different.

For an element which has been subjected to yield, we will assume that the strain will decrease linearly with the load alteration and with unchanged modulus of elasticity, but the zero point is displaced, so strain will still be present when the load has been removed. Therefore, yield will produce internal stresses in the adhesive joint.

With decreasing load on an adhesive joint after yield at the load P_{max}, all stress changes will be elastic at the beginning, and the load change $-(P_{max} - P)$ will produce a stress change distribution that would be the same as if the stresses were acting on a stress-free adhesive joint.

The new stress condition created by the load P is thus determined by adding or superimposing the stresses belonging to the load P_{max} on the stress changes belonging to the load change $-(P_{max} - P)$, which is therefore in the opposite direction.

If the load $-(P_{max} - P)$ can be established solely by elastic stresses, the addition of the two loads is simple. If the load $-(P_{max} - P)$ is so great as normally to cause an opposite-direction yield, *i.e.* a negative yield, it must be realised that $-(P_{max} - P)$ is a fictitious load which does not in itself lead to a yielding even if it shows nominal stresses that are numerically greater than τ_f.

Negative yield will occur only when the actual stresses caused by addition of P_{max} and $-(P_{max} - P)$ become numerically greater than τ_f. In practice, even with complete unloading to $P = 0$, it is possible to get a negative yield only in zones which had positive yield earlier and, thus, the stress τ_f. Therefore, in order to get a negative yield with actual stresses of $-\tau_f$, it is necessary that the superimposed load show negative stresses of $-2\tau_f$ or more. This means that for the superimposed load $-(P_{max} - P)$ we shall calculate with a stress distribution corresponding to a yield stress of $-2\tau_f$.

Negative yields can be handled provided we use the same curves for superimposed loads (Figure 5) as the ones that were used to calculate P_{max}. Now, however, all stresses and loads are twice the magnitude of those given in the figures indicated by the scale for the same length of yield zones. A load with positive yield will, therefore, be followed only with negative yield, when it is unloaded with at least $2P_e$, where P_e is the maximum load attainable without yield. We will give a few examples below.

In Figure 6, curve 1 shows the stress distribution in an adhesive joint with constant elastic stresses τ_f at the ends and with $P_e = 1150$ N. Curve 2 shows the same adhesive joint with yield stress in a longer zone at $P = 1675$ N. Curve 3 shows the load $P = -1675$ N which results from curve 1 by multiplication with $-1675/1150$ giving maximum negative stresses of $-1.46\,\tau_f$; consequently, it is without yield. Curve 4 is the sum of curves 2 and 3 corresponding to full unloading after yield in curve 2. Curve 5 shows the load $P = 1150$ N after the yield. It is the result of adding curves 1 and 4. A comparison of curves 1 and 5 shows that after the yield, stresses have been reduced at the ends and increased in the midsection.

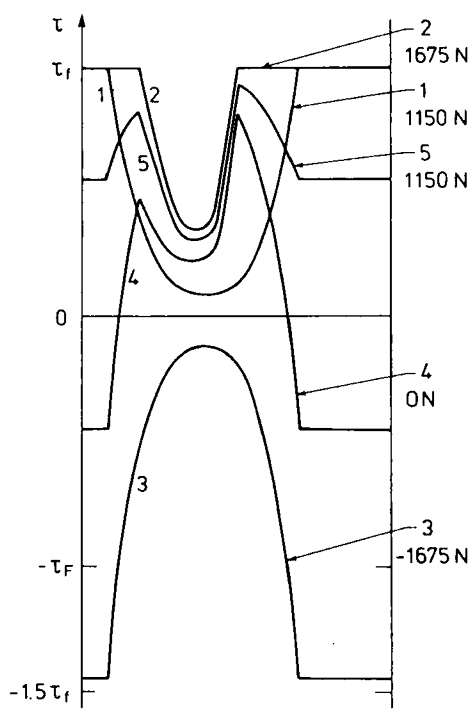

FIGURE 6 1. Stresses prior to yield at 1150 N. 2. Stresses at yield at 1675N. 3. Fictive negative stresses at −1675 N. 4. Full unloading 2 + 3. 5. 1150 N after the yield 1 + 4.

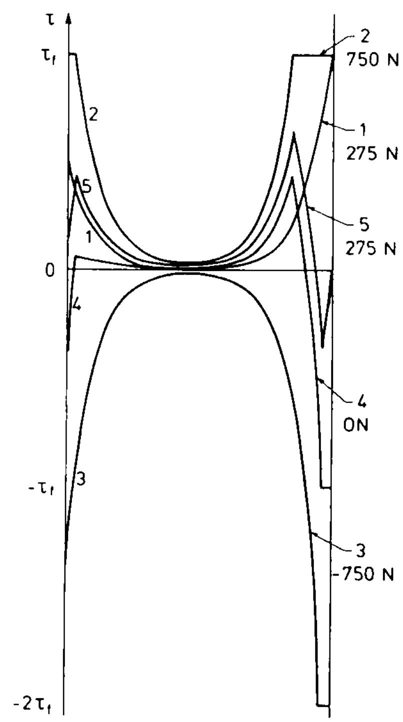

FIGURE 7 1. Stresses prior to yield at 275 N. 2. Stresses at yield at 750 N. 3. Fictive negative stresses at −750 N. 4. Full unloading 2 + 3. 5. 275 N after yield 1 + 4.

In Figure 7, curve 1 shows the maximum stresses in a simple adhesive joint without yield at $P_e = 275$ N. Curve 2 shows the same adhesive joint with yield at $P = 750$ N. Curve 3 shows the stresses for $P = -750$ N which is numerically greater than $2 \cdot 275 = 550$ N and, thus, results in a negative yield with fictitious stresses of $-2\,\tau_f$. The length of the zone is determined by Figure 5 when the stresses and loads are doubled.

This produces real negative yield stresses of $-\tau_f$ in curve 4 which is the sum of curves 2 and 3 corresponding to full unloading. Curve 5 is the sum of curves 1 and 4 corresponding to $P = 275$ N after yielding and unloading. Curves 1 and 5 show that also here the maximum stresses have been reduced and moved inwardly after yield.

Negative yield occurs in a narrow zone only and, consequently, the negative strain is small.

It appears from the method used to determine the negative yield zone that this must always be substantially narrower than the positive zone. It is seen that a controlled overload that moves and reduces the maximum stresses may be an advantage.

We will now design zones with constant elastic stresses.

VARYING THICKNESS OF ADHESIVE

We wish to establish constant elastic shear stresses along the zone b in Figures 1 and 5.

If we demand a constant shear stress τ_f in this zone without yield stress occurring, *i.e.* that all deformations are elastic, then the following will apply to each point in the zone:

$$t_a = \frac{G \cdot \Delta y}{\tau_f} \tag{10}$$

where t_a is variable. We will, therefore, call the constant thickness of the remainder of the adhesive t_{ao}. By introducing Δy from Equation (8) we get:

$$t_a = t_{ao} + \frac{G}{t_1 \cdot E_1}\left(\frac{P \cdot y}{\tau_f} - b \cdot y + \frac{y^2}{2}\right) - \frac{G}{t_2 \cdot E_2} \cdot \left(by - \frac{y^2}{2}\right) \tag{11}$$

which will give constant stress in the joint in the zone b.

τ_f is a limit value, just attainable solely by elastic deformation by action of the force P. However, an increase of P will immediately lead to yield in the whole zone b and in an adjoining zone of the adhesive joint.

So far, we have used the terms b and y for both the yield zone and zone with constant load, because these zones are partly determined by the same formulae. When we will now discuss a yield zone containing a zone designed for constant elastic stress we will distinguish the new and old zones by B, y_B, P_B and b, y_b, P_b, respectively. The initial point for the zone b which has been designed for constant stress, will be termed D.

When P is increased to P_B the result is a longer yield zone B, but its length can be determined exactly as we previously have done for b by means of Figure 5.

Naturally, we are able to make this determination because during yield we have the same stress τ_f both in B and b irrespective of whether or not a varying thickness t_a of the adhesive has been introduced.

Furthermore, this means that the shear of adherend 1 towards the right relative to adherend 2 can be calculated from Eq. (8). After inserting B, y_B and P_B, this equation will become:

$$\Delta y_B = \frac{\tau_f \cdot t_{ao}}{G} + \frac{1}{t_1 \cdot E_1}\left(P_B \cdot y_B - \tau_f \cdot \left(B \cdot y_B - \frac{y_B^2}{2}\right)\right) - \frac{\tau_f}{t_2 \cdot E_2}\left(B \cdot y_B - \frac{y_B^2}{2}\right) \tag{12}$$

The corresponding strain is:

$$\gamma = \frac{\Delta y_B}{t_a}$$

As is shown in curve 1 of Figure 8, Δy_B will increase through the zone, while γ will increase towards the point D and subsequently decrease as t_a grows larger. However, the constancy of shear stress as t_a varies according to Eq. (11) applies fully only as long as no yield occurs in the adhesive joint. This is due to the decrease with increasing adhesive thickness of the ultimate stress of the adhesive

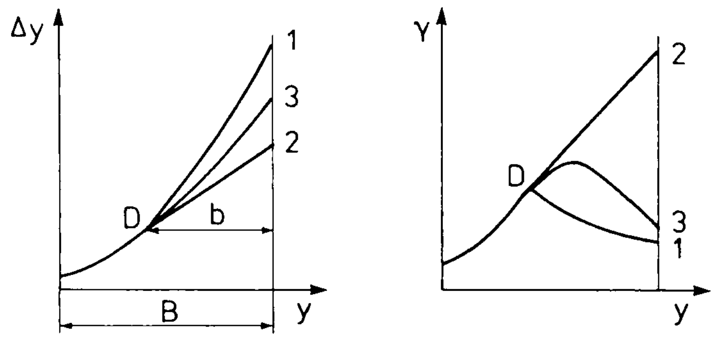

FIGURE 8 Shear and strain in zone B during yield. 1. Varying thickness of adhesive. 2. Varying thickness of adherend. 3. Varying thicknesses of adherend + adhesive.

and its yield stress, which are of almost the same magnitude. This means that the shear stresses at a point in the zone cannot exceed the yield stress corresponding to the adhesive thickness at this point. When yield occurs in the zone, the shear stress will consequently decrease towards the end of the zone with the result that the zone absorbs less shear force than by shear stress, that stays constantly high.

In the following we will disregard the decrease of the yield stress with increasing adhesive thickness; however, the effect can be partly compensated by using a somewhat lower yield stress, *e.g.* corresponding to a mean adhesive thickness in the yield zone.

VARYING THICKNESS OF ADHEREND

Stress equalisation may also be produced with constant adhesive thickness, but varying adherend thicknesses.

If we wish to produce constant elastic shear stress in a zone b of the adhesive joint in this manner, this would require the shear between the adherends 1 and 2 to be the same at all points along the zone. This would mean that the specific extensions of the two adherends are equal at each point of the zone.

In zone b we maintain t_1 constantly equal to t_{10} at the entry to the joint, and the stresses in adherend 1 are given in zone b by Eq. (5), whereby the specific extensions become:

$$\varepsilon_{1y} = \frac{P - \tau_f(b - y)}{t_{10} \cdot E_1} \tag{13}$$

In adherend 2 we will let the thickness t_2 vary. The stresses are given by Eq. (3), and the specific extensions will become:

$$\varepsilon_{2y} = \frac{\tau_f}{t_2 \cdot E_2} \cdot (b - y) \tag{14}$$

we set $\varepsilon_{1y} = \varepsilon_{2y}$ and get:

$$t_2 = \frac{t_{10} \cdot E_1 \cdot \tau_f \cdot (b-y)}{E_2 \cdot (\ P - \tau_f \cdot (b-y))} = \frac{t_{20}}{c_3} \cdot \frac{\tau_f \cdot (b-y)}{P - \tau_f (b-y)} \tag{15}$$

which results in constant shear stress in zone b.

Normally, the value of t_2 determined by Eq. (15) for $y = 0$ will differ from the constant thickness t_{20} throughout the rest of the adhesive joint. This means that a step in the thickness t_2 is needed at the transition from varying to constant stress. When the force increases from P to P_B we will, as before, get yield in a zone B which is greater than b. Along the initial length of B to the point D, t_2 is constantly equal to t_{20} and again Δy_B is determined by Eq. (12). For zone b we will determine the length extension of the adherends 1 and 2 separately in relation to the point D which here is taken to be the zero point for the y-axis.

The force is adherend 1 increases linearly from $(P_B - \tau_f \cdot b)$ in D to P_B for $y = b$:

$$P_{1y} = P_B - \tau_f \cdot (b-y)$$

which, corresponding to Eq. (6), will give:

$$\Delta y_1 = \frac{1}{t_{10} \cdot E_1} \left(P_B \cdot y - \tau_f \cdot b \cdot y + \tau_f \cdot \frac{y^2}{2} \right)$$

The force in adherend 2 will decrease linearly from $b \cdot \tau_f$ in D to 0 for $y = b$:

$$P_{2y} = \tau_f \cdot (b-y)$$

and since t_2 is varying we will get, corresponding to Eq. (4):

$$\Delta y_2 = \frac{\tau_f}{E_2} \cdot \int_o^y \frac{b-y}{t_2}\, dy$$

where we introduce t_2 from Eq. (15):

$$\Delta y_2 = \frac{\tau_f}{E_2} \int_o^y \frac{E_2 \cdot (P - \tau_f \cdot (b-y))}{t_{10} \cdot E_1 \cdot \tau_f \cdot (b-y)} \cdot (b-y)\, dy$$

$$\Delta y_2 = \frac{1}{t_{10} \cdot E_1} \cdot \int_o^y (P - \tau_f \cdot (b-y))\, dy$$

$$\Delta y_2 = \frac{1}{t_{10} \cdot E_1} \left(P \cdot y - \tau_f \cdot b \cdot y + \tau_f \cdot \frac{y^2}{2} \right)$$

The shear in zone b is then:

$$\Delta y = \Delta y_D + \Delta y_1 - \Delta y_2 = \Delta y_D + \frac{(P_B - P) \cdot y}{t_{10} \cdot E_1}$$

Actually, this shear increases linearly with y, and the same applies to the strain

$$\gamma = \frac{\Delta y}{t_a}$$

As shown in Figure 8, curve 2, this contrasts with decreasing γ at varying t_a.

A step in the thickness t_2 is avoidable in only one case, *viz.* when the zone b becomes so long that it meets the corresponding zone a in adherend 1 calculated to have the same constant stress, whereby the complete adhesive joint gets constant stress. We will determine a and b for this case.

In the point where a and b meet we have:

$$\varepsilon_1 = \varepsilon_2 = \frac{a \cdot \tau_f}{t_{10} \cdot E_1} = \frac{b \cdot \tau_f}{t_{20} \cdot E_2}$$

$$\frac{b}{a} = \frac{t_{20} \cdot E_2}{t_{10} \cdot E_1} = c_3 \qquad a + b = L \tag{16}$$

$$b = L \cdot \frac{c_3}{1 + c_3} \qquad a = \frac{L}{1 + c_3}$$

corresponding to Eq. (15), the varying thickness of adherend 1 is determined by:

$$t_1 = \frac{t_{20} \cdot E_2 \cdot \tau_f(a - y)}{E_1 \cdot (P - \tau_f \cdot (a - y))} = c_3 \cdot t_{10} \cdot \frac{\tau_f \cdot (a - y)}{P - \tau_f \cdot (a - y)} \tag{17}$$

in which case the y-axis is considered positive towards the left. On differentiating Eqs. (15) and (17) we find that the inclination at the ends of adherends 1 and 2 is determined by:

$$tg\alpha_1 = c_3 \cdot t_{10} \cdot \frac{\tau_f}{P} \tag{18}$$

$$tg\alpha_2 = \frac{t_{20}}{c_3} \cdot \frac{\tau_f}{P} \tag{19}$$

It is seen that the inclinations are inversely proportional to P and are thus minimum for constant stress throughout the length of of the joint. In this case we have:

$$P = \tau_f \cdot L \qquad tg\alpha_1 = c_3 \cdot \frac{t_{10}}{L} \qquad tg\alpha_2 = \frac{t_{20}}{c_3 \cdot L}$$

Tensile stress at the end of an adherend will be determined by:

$$\sigma = \frac{\tau \cdot dy}{tg\alpha \cdot dy} = \frac{\tau}{tg\alpha} \tag{20}$$

$$\sigma_1 = \frac{\tau_f}{tg\alpha_1} = \frac{1}{c_3} \cdot \frac{P}{t_{10}} = \frac{1}{c_3} \cdot \sigma_{10} \tag{21}$$

$$\sigma_2 = \frac{\tau_f}{tg\alpha_2} = c_3 \cdot \frac{P}{t_{20}} = c_3 \cdot \sigma_{20} \tag{22}$$

where σ_{10} and σ_{20} are the stresses in the adherends where they enter the adhesive joint.

Normally, adhesive joints are dimensioned so that it is not the joint but one or both of the adjoining adherends which are the weakest link with the stresses σ_{10} and σ_{20}. Therefore, it may prove adverse if, at the end of an adherend, greater stresses such as σ_1 or σ_2 are present which will then cause yield in the adherend. The yield will be minor only, without much significance to static load, but it should be avoided for dynamic load.

For $c_3 = 1$, we have $\sigma_1 = \sigma_{10}$ and $\sigma_2 = \sigma_{20}$, and there is no problem. It is seen that only the rigid adherend with the stress σ_2 may cause problems. Bonding of two adherends of the same material causes no problems, since here we get

$$\sigma_2 = c_3 \cdot \sigma_{20} = \frac{t_2}{t_1} \cdot \sigma_{20} = \sigma_{10}$$

If we will demand that nowhere in adherend 2 will the stress σ exceed the yield stress σ_{2f} for $\tau = \tau_f$, the thickness of adherend 2 shall everywhere in the zone be equal to or exceed the thicknesses indicated by a straight line from the end of adherend 2 with an inclination of:

$$tg\alpha = \frac{\tau_f}{\sigma_{2f}}$$

Constant shear stress along the complete adhesive joint is also attainable by simultaneously varying the thicknesses of both adherends along the full length of the adhesive joint, and this is achievable in many ways. The general conditions are that the specific extensions of the two adherends are equal at any point of the joint:

$$\varepsilon_{1x} = \frac{x \cdot \tau_f}{t_1 \cdot E_1} \qquad \varepsilon_{2x} = (L - x) \cdot \frac{\tau_f}{t_2 \cdot E_2}$$

$$t_2 = t_1 \cdot \frac{E_1}{E_2} \cdot \frac{L - x}{x}$$

A linear variation of t_1 from 0 to t_{1L} gives a linear variation of t_2:

$$t_1 = \frac{x}{L} \cdot t_{1L} \qquad t_2 = t_{1L} \cdot \frac{E_1}{E_2} \cdot \frac{L - x}{L}$$

for $x = 0$:

$$t_{20} = t_{1L} \cdot \frac{E_1}{E_2} \qquad t_{20} \cdot E_2 = t_{1L} \cdot E_1 \tag{23}$$

Here t_{1L} corresponds to the previously used term t_{10} because then we calculated the y-axis with opposite direction for adherend 1.

Equation (23) then gives $c_3 = 1$ and, if the adherends do not have that value, there will be a step in the thickness at the end of the joint.

VARYING THICKNESSES OF BOTH ADHEREND AND ADHESIVE

To avoid a yield stress in an adherend and get a more robust termination of an adherend, it may be advantageous to use a combination of varying thicknesses of both adherends and adhesive. A variation of adherend thickness is chosen which approximates as much as possible the variation corresponding to constant adhesive thickness, but without obvious inconveniences.

We set $t_2 = f(y)$ and get

$$\sigma_{2y} = \frac{\tau_f \cdot (b-y)}{t_2} \qquad \Delta y_2 = \int_o^y \frac{\sigma_{2y}}{E_2}\,dy = \frac{\tau_f}{E_2} \cdot \int_o^y \frac{b-y}{t_2}\,dy$$

The extension of adherend 1 is given by Eq. (6), and by using Eq. (7) we get:

$$\Delta y = \frac{\tau_f \cdot t_{ao}}{G} + \frac{1}{t_1 \cdot E_1}\left(P \cdot y - \tau_f\left(by - \frac{y^2}{2}\right)\right) - \frac{\tau_f}{E_2} \cdot \int_o^y \frac{b-y}{t_2}\,dy \tag{24}$$

By using Eq. (10) we get:

$$t_a = t_{ao} + \frac{G}{t_1 \cdot E_1}\left(\frac{P \cdot y}{\tau_f} - b \cdot y + \frac{y^2}{2}\right) - \frac{G}{E_2} \cdot \int_o^y \frac{b-y}{t_2} \cdot dy \tag{25}$$

which gives the correlation between t_a and t_2 for constant elastic stress in the zone b.

In Figure 9 the curves 1, 2, 3 and 4 show correlated variations of t_a and t_2. The area below the t_a-curve is an expression of the effect of the corresponding variation of t_2, inasmuch as a small area implies efficient adaption. It is seen that a straight inclined cut-off (curve 2) is almost without effect as the t_a-area is almost the same as for curve 1 which completely lacks any variation for t_2. Curve 3 shows some effect, but curve 4 shows that here also a step is useful in t_2 as was the case for the variation of t_2 alone.

Figure 10 shows the curves 4 in true scale. The yield in a joint after the curves marked 4 is shown as curve 3 in Figure 8.

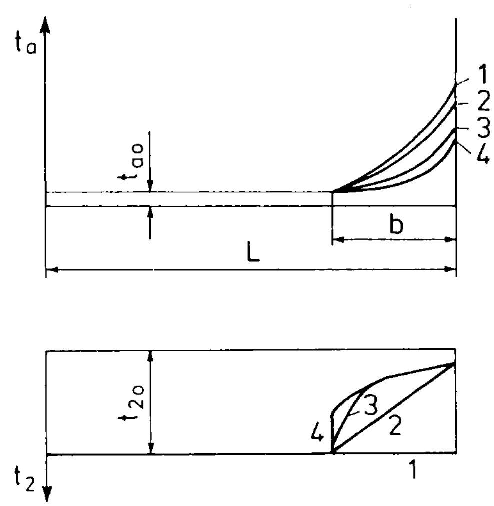

FIGURE 9 Associated curves 1, 2, 3, and 4 for varying adhesive thicknesses (top) and varying adherend thicknesses (bottom).

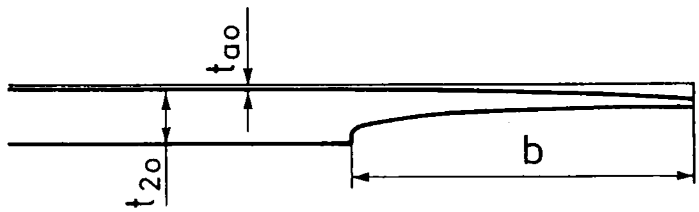

FIGURE 10 Drawing, to scale, of zone with varying adherend + adhesive according to the two curves marked 4 in Figure 9.

Finally, we will look into the case of constant elastic shear stress along the full length of the joint obtained by simultaneously varying t_1, t_2 and t_a:

$$\sigma_{1x} = \frac{\tau_f \cdot x}{t_1} \qquad \sigma_{2x} = \frac{\tau_f \cdot (L-x)}{t_2}$$

$$\Delta x_1 = \int_o^x \frac{\tau_f \cdot x}{E_1 \cdot t_1} dx \qquad \Delta x_2 = \int_o^x \frac{\tau_f \cdot (L-x)}{E_2 \cdot t_2} dx$$

$$\Delta x = \frac{\tau_f \cdot t_{ao}}{G} + \int_o^x \frac{\tau_f \cdot x}{E_1 \cdot t_1} \cdot dx - \int_o^x \frac{\tau_f \cdot (L-x)}{E_2 \cdot t_2} dx$$

To get constant elastic shear stress τ_f:

$$t_a = \frac{G \cdot \Delta x}{\tau_f}$$

$$t_a = t_{ao} + \frac{G}{E_1} \cdot \int_o^x \frac{x}{t_1} d_x - \frac{G}{E_2} \int_o^x \frac{L-x}{t_2} dx$$

ACCURATE CALCULATION OF ZONES WITH CONSTANT SHEAR STRESSES

Zones with constant elastic shear stresses can be designed taking all relevant factors into consideration including various temperature-induced stresses, bending moments in the adherends, etc. As a basis we can use a design of the zones applying one of the methods given above, *e.g.* with constant thickness of adhesive and varying thicknesses of adherends.

With the load P we will then have the shear stress τ_f in the zones, and we also know the stresses and shears between the zones. We can maintain this known distribution of shear stresses also when a more accurate calculation shows altered shear. It can be done when we multiply the thickness of the adhesive in every point of the joint by the ratio of the new shear to the initial shear. Therefore, we will stress each adherend with the force P and the initial distribution of shear stresses.

We will ignore the extension of the adhesive across the joint and thereby assume that the two adherends have exactly the same movement transverse to the joint and thereby the same curvature.

With the relevant factors taken into consideration we can calculate the corrected extensions Δx_{1K} and Δx_{2K} of the adherends from the specific extensions in all points along the joint. The two adherends with the corrected extensions can be placed arbitrary to each other, as it will always be possible to determine the thicknesses of the adhesive in order to get the same distribution of shear stresses as for the basis. We choose the corrected shear Δx_K in one point of the joint, and all the corrected extensions and shears can be found adding to the basic figures the changes produced by taking more factors into consideration. The effect of each of these factors can be calculated separately.

It is remarkable that by calculating the changes of deformation made by taking a new factor into consideration, we will get no contribution from shear stresses in the adhesive. These shear stresses are the same before and after the factor is taken into consideration, because we simultaneously increase the thickness of the adhesive in proportion to the shear. The change is therefore calculated with the shear stress O in the whole length of the adhesive.

At first we will determine the effect of the most important of these factors, namely bending moments in the adherends. We will calculate the common curvature for the adherends in an arbitrary cross section of the joint. The surfaces of the adherends next to the adhesive are exactly those that have a common curvature, but as the radius of curvature is large compared to the thicknesses of adhesive and adherends, we will calculate by taking a curvature for the neutral lines of the adherends.

The bending moment M in such a cross section is the total moment of the external forces and moments round the neutral centre of the section which is calculated as the common center of gravity for 1 and 2, but first the area of 2 shall be multiplied by the factor $\sigma_2 : \sigma_1$. A force working at this point will be in balance with the calculated stresses of the section and, therefore, it will not cause any bending.

As there are no shear forces in the adhesive the moment will be taken up by the two adherends separately, but with a common radius of curvature r:

$$\frac{1}{r} = \frac{M}{E_1 \cdot I_1 + E_2 \cdot I_2} = \frac{12M}{E_1 \cdot t_1^3 + E_2 \cdot t_2^3}$$

We consider the curvature as positive when the centre of curvature is above the joint (Figure 1) and moments are positive when they make positive curvatures. The specific extensions of the surfaces next to the adhesive will be:

$$\varepsilon_{1M} = \frac{t_1}{2r} = \frac{6 \cdot t_1 \cdot M}{E_1 \cdot t_1^3 + E_2 \cdot t_2^3} \qquad \varepsilon_{2M} = -\frac{6 \cdot t_2 \cdot M}{E_1 \cdot t_1^3 + E_2 \cdot t_2^3}$$

and the specific shear defined as $\varepsilon_M = \varepsilon_{1M} - {}_{2M}$ will be:

$$\varepsilon_M = \frac{6M \cdot (t_1 + t_2)}{E_1 \cdot t_1^3 + E_2 \cdot t^3}$$

After determining the specific shears at some points of the joint, we can draw a curve for ε_M as a function of x, and the shear Δx_M in the point x can be found by:

$$\Delta x_M = \int_{1/2L}^{x} \varepsilon_M \, dx = \int_{1/2L}^{x} \frac{6M \cdot (t_1 + t_2)}{E_1 \cdot t_1^3 + E_2 \cdot t_2^3} \, dx$$

We chose the mid-point of the joint $x = \frac{1}{2} L$ here as an initial point for all extensions, and for the same point we chose to keep the shear unchanged at $\Delta x_K = \Delta x$ and thereby keep the thickness of the adhesive $t_{ak} = t_a$ unchanged as well. The choice will determine the absolute sizes of the shear and adhesive thickness. If we have constant shear stresses in the whole joint it may be advantageous to choose as initial point the place where the zones a and b meet say $x = a$, and make this the lower limit for the integration.

We will now consider a linear variation of temperature across the joint.

In a cross section at right angles to the joint, the temperature varies linearly from T_{1t} on the top side of adherend 1 to T_{1u} on the under side, and further from T_{2t} to T_{2u} through adherend 2. If the adherends could move freely, by means of the coefficients of thermal extension α_1 and α_2, we would get angles $d\beta_1$ and $d\beta_2$ between the main section and the sections I and II that were previously parallel to the main section in the distance dx:

$$d\beta_1 = \alpha_1 \cdot \frac{T_{1u} - T_{1t}}{t_1} \cdot dx \qquad d\beta_2 = \alpha_2 \cdot \frac{T_{2u} - T_{2t}}{t_2} \cdot dx$$

A common angular turn $d\beta$ for these two sections will lie between $d\beta_1$ and $d\beta_2$ with deviations from them inversely proportional with $E.I = 1/12 \cdot E \cdot t^3$

$$d\beta = d\beta_1 + (d\beta_2 - d\beta_1) \cdot \frac{E_2 \cdot I_2}{E_1 \cdot I_1 + E_2 \cdot I_2}$$

After some calculation the specific turn calculated for the length 1 of the joint comes out to be:

$$\beta = \frac{\alpha_1 \cdot (T_{1u} - T_{1t}) \cdot E_1 \cdot t_1^2 + \alpha_2 (T_{2u} - T_{2t}) \cdot E_2 \cdot t_2^2}{E_1 \cdot t_1^3 + E_2 \cdot t_2^3}$$

The specific extensions of the surfaces of the adherends next to the adhesive will be:

$$\varepsilon_{1T} = \tfrac{1}{2} t_1 \cdot \beta \qquad \varepsilon_{2T} = -\tfrac{1}{2} t_2 \cdot \beta$$

and the specific shear:

$$\varepsilon_T = \varepsilon_{1T} - \varepsilon_{2T} = \tfrac{1}{2}(t_1 + t_2) \cdot \beta$$

giving the shear:

$$\Delta x_T = \int_{1/2L}^{x} 1/2(t_1 + t_2) \cdot \beta \cdot dx$$

after a curve has been drawn for $\frac{1}{2}(t_1 + t_2) \cdot \beta$ as a function of x. There will also be a direct effect of the temperatures T_{1u} and T_{2t} of the surfaces next to the adhesive.

They will normally vary along the joint due to the varying thicknesses of the adherends, and they will differ from the hardening temperature T_h for the adhesive. The resulting specific extensions are:

$$\varepsilon_{1Th} = \alpha_1(T_{1u} - T_h) \qquad \varepsilon_{2Th} = \alpha_2(T_{2t} - T_h)$$

and the shear:

$$\Delta x_{Th} = \int_{1/2L}^{x} (\alpha_1 \cdot (T_{1u} - T_h) - \alpha_2(T_{2t} - T_h))\, dx$$

Even for the case that the whole joint has the constant temperature T these will contribute to the shear with

$$\Delta x_{ThW} = (\alpha_1 - \alpha_2) \cdot (T - T_h) \cdot (x - \tfrac{1}{2}L)$$

When the effect of bending moments and temperature variations are added to the basic load, we get the corrected shears:

$$\Delta x_K = \Delta x + \Delta x_M + \Delta x_T + \Delta x_{Th}$$

The corrected thicknesses of the adhesive for maintaining the initial distribution of shear stresses in the adhesive will be:

$$t_{ak} = t_a \cdot \frac{\Delta x_K}{\Delta x}$$

The design of the zone with constant elastic shear stresses, taking into consideration bending moments, will be generally usable also for single lab joints, and stresses will remain constant as long as no yield takes place.

The correction for temperature variations will give only constant stresses for the load for which it is calculated.

Should greater accuracy be needed, we could change the assumption that the movement is the same in a direction transverse to the joint, and instead assume that the distance between the adherends will be determined by the transverse extensions of the adhesive using the thicknesses t_{ak}. For such calculations a finite element method will be used and thereby it will also be possible to consider shear in the adherends.

As we will get varying thickness of adhesive in the whole joint, it can be practical also to apply constant shear stresses along the whole length of the adhesive. We thereby also avoid making a curve as Figure 5, as we then directly have $P = L \cdot \tau_f$ and $b : a = c_3$.

USE OF CONSTANT ELASTIC SHEAR STRESS

Hart-Smith recommends in Ref. 1 that the lowest stress of the adhesive should be at most 10% of the maximum stress. This will result in a depression or "trough" in the stress curve with low stresses in the midsection of the joint, and he has had good results with this. Hart-Smith assumes that the advantage of a deep trough may be that the negative yield stress thus created may prevent accumulated yield with varying loads. We have seen that negative yield occurs rather rarely and only to such a limited extent that it cannot have much positive effect.

On the other hand, we see that in zones with negative yield we will get alternating positive and negative yield by loading and unloading.

When negative yield is avoided we will get yield at the initial loading only, while at the subsequent loadings of the same magnitude we just reach the yield stress. After a small accidental overloading we can even avoid yield stress at normal load.

Consequently, it must be advantageous to avoid negative yield, and for alternating load O–P this is feasiable by keeping $P < 2P_e$. For alternating $\pm P$ the load change is $2P$. Positive and negative yields are equal, and yield should be totally avoided.

The effect of a deep trough must be due to other factors, and here it is worth mentioning that such a trough has the important property that a certain increase in load will result in a relatively small increase of the length of the yield zone and of the strain at the end of the adhesive joint, while a shallow trough results in large increase of these quantities. This may be of special importance if an adhesive has a stress-strain curve which shows decreasing yield stress at increased strain after a maximum is reached. Linearly decreasing yield stress in adhesives is discussed in Ref. 3.

As the load increases, the yield zone will increase. With such an adhesive and a shallow trough it may happen, that an increase in the length of the yield zone cannot offset the loss from decreasing stress at the end of the zone due to increased strain. This will cause a fracture. With a large "trough" this effect occurs later or not at all, and we can get a longer yield zone and a higher load.

It will be possible to combine 10% minimal stresses with zones with constant elastic shear stress, but in this connection it is preferable that these zones have such an extension that yield is completely avoided. In that case, joints with constant stress along the complete adhesive joint also will be usable; it is only a matter of having a sufficient safety factor.

Zones with constant stresses without yield have the added advantage that their calculation is wholly independent of the actual course of the stress-strain curve of the adhesive as regards yield stress and strain, the knowledge of the correlation of which is frequently insufficient. It is well-known that simple adhesive joints without yield rather soon reach a length where the load cannot be increased more if the length of the adhesive joint is increased. Something similar applies to these when only a certain limited strain is allowed.

In contrast, the maximum load on an adhesive joint with constant elastic stress along the whole length will be proportional to the length of the adhesive joint.

Actually, by selecting the length of the adhesive joint it is possible to keep the shear stress of the adhesive below a certain desirable value in this case, and in this way choose the safety factor required. To avoid an end of the most stiff adherend that is too thin or of the adhesive that is too thick, it can be advantageous to use a combined variation of the thickness of adherends and adhesive. Thereby, we get a favourable stress distribution in the adherends and a practical design of both adherends and adhesive. At the same time, we have the advantage of reduced peel-stresses due to the variation of thickness of adherends, and if yield occurs we get reduced strain at the end of the adhesive due to the variation of the thickness of the adhesive.

It will always be advantageous to strengthen a simple adhesive joint by establishing constant elastic shear stress at the ends, and for shorter zones a linear variation of the thickness of the adhesive and/or the adherends will be a good approximation. For a variation of the adherends a step in the thickness is required at the beginning of the zone, but the variation of thickness of adhesive is made without a step and requires very little machining; at the same time this makes the joint less sensitive to inaccuracies.

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